MIT OCW GR 8.962 PSET 11

1 · Formation of a black hole

· In this problem we will see how a schwarzchild black hole can be formed from the collapse of a simple, non-singular physical object:

· The exterior of a star of pressureless dust is described by:

$$ds^2_{r>R_*} = -\left(1-\frac{2GM}{r}\right)dt^2 + \frac{dr^2}{1-2GM/r} + r^2 d\Omega^2$$

· The interior is described by:

$$ds^2_{r<R_*} = -d\tau^2 + a^2(\tau)R_0^2\left(d\chi^2 + \sin^2\chi\, d\Omega^2\right)$$

a · Show that the parametric solution:

$$a = \frac{a_{max}}{2}(1+\cos\eta)$$

$$\tau = \frac{a_{max}R_0}{2}(\eta + \sin\eta)$$

with $0 \le \eta \le \pi$ solves the Friedmann Equations for $k=1$ with ρ given by the energy density for pressureless dust matter:

$$(F1) \quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3}$$

$$(F2) \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3}$$

· Assume $\Lambda \approx P \approx 0$ and $\rho = \rho_0 a^{-3}$ for a matter dominated energy density:

$$\rightarrow \left\{ \begin{array}{l} \left(\frac{\dot{a}}{a}\right)^2 = c_0 a^{-3} - a^{-2} \quad (\star) \\ \left(\frac{\ddot{a}}{a}\right) = \frac{-c_0 a^{-3}}{2} \quad \text{where } c_0 \equiv \frac{8\pi G\rho_0}{3} \end{array} \right\}$$

· We will now show that the parametric solution obeys the first Friedmann equation $(\star)$

· Find $\dot{a} = \frac{da}{d\tau} = \frac{da}{d\eta} \cdot \frac{d\eta}{d\tau}$:

· First $d\eta / d\tau$:

$$\frac{d}{d\tau}(\tau) = \frac{d}{d\tau}\left(\frac{a_m R_0}{2}(\eta + \sin\eta)\right)$$

$$\rightarrow 1 = \left(\frac{a_m R_0}{2} \cdot \frac{d\eta}{d\tau}\right)(1 + \cos\eta)$$

$$\rightarrow \frac{d\eta}{d\tau} = 2 / (a_m R_0)(1 + \cos\eta)$$

• <u>Now</u> find $da/d\eta$:

$$\frac{da}{d\eta} = \frac{d}{d\eta}\left(\frac{a_m}{2}(1+\cos\eta)\right) = \frac{-a_m \sin\eta}{2}$$

$$\rightarrow \frac{da}{d\tau} \equiv \dot{a} = \frac{-\sin(\eta)}{R_o(1+\cos\eta)}$$

$$\rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{\sin^2(\eta)}{R_o^2(1+\cos\eta)^2} \cdot \frac{4}{a_m^2(1+\cos\eta)^2}$$

$$\rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{4\sin^2\eta}{R_o^2(1+\cos\eta)^4} \quad \text{if } a_m = 1 \text{ by construction}$$

• Now check does this obey $\left(\frac{\dot{a}}{a}\right)^2 = C_o a^{-3} - a^{-2}$?

$$\text{RHS} = C_o a^{-3} - a^{-2}$$

$$= \frac{C_o 2^3}{(1+\cos\eta)^3} - \frac{2^2}{(1+\cos\eta)^2} \quad \text{if } a_m = 1 \ldots$$

$$= \frac{4}{R_o^2(1+\cos\eta)^4} \cdot \underbrace{\left(2C_o R_o^2(1+\cos\eta) - R_o^2(1+\cos\eta)^2\right)}$$

• So we need this term $(\sim\sim)$ $= \sin^2\eta$ for a match: $\longrightarrow$

• So let's equate the 2 and <u>solve</u>:

$$R_0^2\left(2c_0(1+\cos\eta)-(1+2\cos\eta+\cos^2\eta)\right)=\sin^2\eta$$

$$R_0^2\left(2c_0(1+\cos\eta)-2(1+\cos\eta)+\sin^2\eta\right)=\sin^2\eta$$

• If R_0 was equal to 1, we could just set $c_0=1$, but we cannot in general:

$$\rightarrow c_0=\frac{(\sin^2\eta)(1-R_0^2)}{(2R_0^2)(1+\cos\eta)}+1 \quad (*)$$

• If (*) is the case, then the parametric equations satisfy the 1st Friedmann equation. The 2nd Friedmann equation is derived from F1 + energy conservation so if all parameters are tuned correctly; the F2 equation should be satisfied as well ✓✓

• since $c_0=8\pi G\rho_0/3$ we can use (*) to derive a relationship between initial density ρ_0 and the length-scale R_0:

$$\boxed{\rho_0=\left(\frac{3}{8\pi G}\right)\left(\frac{(\sin^2\eta)(1-R_0^2)}{(2R_0^2)(1+\cos\eta)}+1\right)}$$

6 . The solution for the interior time coordinate τ is only good up to $\tau = \pi R_0/2$. What happens to the interior solution after that?

$$\tau = \frac{\pi \cancel{R_0}}{\cancel{2}} = \frac{\cancel{R_0}}{\cancel{2}}(\eta + \sin\eta) \rightarrow \eta + \sin\eta = \pi$$

• Plotting this transcendental equation, one finds that: $\eta = \pi$. Now plug this into $\dot{a}$ formula:

$$\dot{a} = \frac{-\sin(\pi)}{R_0(1+\cos(\pi))} \rightarrow \frac{0}{0}$$ Now use L'Hopital's rule:

$$\lim_{\eta \to \pi} \dot{a} = \frac{-\cos(\eta)}{-\sin(\eta)} \rightarrow \frac{+1}{-0} \rightarrow \infty$$

• So as $\tau \rightarrow \pi R_0/2$, $\dot{a} \rightarrow \infty$ implying the scale factor grows without bound + there is a "Big Rip" inside the star. This is unphysical so the lesson is our solution breaks down as $\tau \rightarrow \pi R_0/2$ ... ✓

2 . Consider a purely radial "orbit" (i.e. trajectory with no angular momentum $L = 0$). For a given energy per unit mass E, find the radius R at which the radial velocity goes to zero:

• We will use this solution to define the "orbital energy" of a dust element at the surface of the star as it begins to collapse:

- The 4-mom. of a particle at the surface is:

$$\vec{P} = p^{\nu} = m\left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\varphi}{d\tau}\right)$$

- If $\vec{L} = 0$ this means $\frac{d\theta}{d\tau} = \frac{d\varphi}{d\tau} = 0$

$$\rightarrow p^{\nu} = m\left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, 0\right)$$

- Remember at the surface of the star the metric is given by the Schwarzschild line element:

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \frac{dr^2}{\left(1-\frac{2GM}{r}\right)} + \underbrace{r^2 d\Omega^2}_{\rightarrow 0 \text{ since } L=0}$$

- Calculate $p_t = -E = g_{tt}\, p^t$

$$-E = -\left(1-\frac{2GM}{r}\right) m \frac{dt}{d\tau} \longrightarrow \frac{dt}{d\tau} = \frac{E/m}{(1-2GM/r)}$$

$$\rightarrow \frac{dt}{d\tau} = \frac{\hat{E}}{(1-2GM/r)} \quad \text{where } \hat{E} \equiv E/m$$

- **Now** calculate $p_r = g_{rr}\, p^r = \frac{m\, dr/d\tau}{(1-2GM/r)}$

- **Now** find $-m^2 = \vec{p}\cdot\vec{p} = g_{\alpha\beta}\, p^{\alpha} p^{\beta}$

$$= p_t p^t + p_r p^r \longrightarrow$$

$$\Rightarrow -m^2 = -E \cdot \frac{m\,dt}{d\tau} + p_r \frac{m\,dr}{d\tau}$$

$$= \frac{-E^2}{(1-2GM/r)} + \frac{m^2 dr^2/d\tau^2}{(1-2GM/r)}$$

$$\rightarrow \frac{m^2 dr^2}{d\tau^2} = E^2 - m^2(1-2GM/r)$$

$$\rightarrow \frac{dr^2}{d\tau^2} = \hat{E}^2 - (1-2GM/r)$$

$$\rightarrow$$

$$0 = \left.\frac{dr}{d\tau}\right|_R = \sqrt{\hat{E}^2 - (1-2GM/R)}$$

$$\rightarrow \boxed{R = \frac{2GM}{1-\hat{E}^2}}$$ defines the radius where $dr/d\tau \to 0$ for a particle with $E/m \equiv \hat{E}$

[d] · Using the radial geodesic equation for the Scwarzchild geometry and the relationship you just found for R and $\hat{E}$ write down an integral for the proper time τ it takes for a fluid element at the star's surface to fall from R_* to r:

• The total proper time to fall from R_* to r is given by:

$$\tau = -\int_{R_*}^{r} \frac{dr'}{dr'/d\tau}$$

• In the last calculation we found that:

$$\frac{dr'}{d\tau} = \sqrt{\hat{E}^2(r') - (1 - 2GM/r')}$$

• If we let $\hat{E}^2(r' = R)$ then:

$$R = \frac{2GM}{1-\hat{E}^2} \longrightarrow \hat{E}^2 = 1 - \frac{2GM}{R_*}$$

• Plug this back into $dr'/d\tau$ to find:

$$\frac{dr'}{d\tau} = \sqrt{\frac{2GM}{r'} - \frac{2GM}{R_*}}$$

$$\longrightarrow \boxed{\tau = -\int_{R_*}^{r} \frac{dr'}{\sqrt{2GM/r' - 2GM/R_*}}}$$

• By introducing the parameterization $r = \frac{R_*}{2}(1 + \cos\eta)$ show that this integral can be evaluated to yield:

$$\tau = \sqrt{\frac{R_*^3}{8GM}}(\eta + \sin\eta)$$

$$\rightarrow dr = \frac{-R_*}{2}\sin(\eta)d\eta$$

$$\rightarrow \tau = \int_{\cos^{-1}(1)}^{\cos^{-1}(2r/R_* - 1)} \frac{\frac{R_*}{2}\sin(\eta)d\eta}{\sqrt{2GM}\sqrt{\frac{2}{R_*(1+\cos\eta)} - 1}}$$

$$\tau = \sqrt{\frac{R_*}{2GM}}\left(\frac{R_*}{2}\right)\int_{\cos^{-1}(1)}^{\cos^{-1}(2r/R_* - 1)} \left(\frac{1+\cos\eta}{1-\cos\eta}\right)^{1/2}\sin(\eta)\,d\eta$$

- Now let $u = \cos(\eta)$, $du = -\sin(\eta)d\eta$

$$\rightarrow \tau = \sqrt{\frac{R_*^3}{8GM}}\int_1^{\cos(\eta)} -\left(\frac{1+u}{1-u}\right)^{1/2} du$$

Now just use an integral calculator...

$$\tau = \sqrt{\frac{R_*^3}{8GM}}\left(2\sin^{-1}\left(\left(\frac{1-u}{2}\right)^{1/2}\right) + \sin\left(2\sin^{-1}\left(\left(\frac{1-u}{2}\right)^{1/2}\right)\right)\right)\Bigg|_1^{\cos\eta}$$

$$\rightarrow \tau = \sqrt{\frac{R_*^3}{8GM}}\left(2\sin^{-1}\left(\sqrt{\frac{1-\cos\eta}{2}}\right) + \sin\left(2\sin^{-1}\left(\sqrt{\frac{1-\cos\eta}{2}}\right)\right)\right.$$

$$\left. - \underbrace{2\sin^{-1}(0)}_{0} + \underbrace{\sin(2\sin^{-1}(0))}_{0}\right)$$

- Use the trig. identity that $\sin\left(\frac{\eta}{2}\right) = \sqrt{\frac{1-\cos\eta}{2}}$

$$\rightarrow \boxed{\tau = \sqrt{\frac{R_*^3}{8GM}}\left(\eta + \sin\eta\right)}$$ as we wanted to show

$\boxed{e}$ [redacted] · We now match the inner and the outer coordinate systems: We require that the star's circumference be the same in both coords for all η and we require the two expressions for proper time τ experienced by a fluid element on the star's surface be the same for all η.

- By enforcing these 2 conditions, determine the lengthscale R_0 and the Robertson-Walker radius of the star χ:

· Given 2 equations for τ:

$$\tau = \frac{R_0}{2}(\eta + \sin\eta)$$

$$\tau = \sqrt{R_*^3 / 8GM}\,(\eta + \sin\eta)$$

· Equate the two:

$$\longrightarrow \boxed{R_0 = \sqrt{R_*^3 / 2GM}} \quad \swarrow$$

An equation relating <u>constants</u> w.r.t. time

· Now equate the circumference in the 2 different coordinate systems:

$$\text{circumference} = 2\pi r(\eta) = 2\pi a(\eta) R_0 \sin(\chi)$$

$$\longrightarrow \chi = \sin^{-1}\left(r / a(\eta) R_0\right)$$

· plug in $R_0 = \sqrt{R_*^3 / 2GM}$ and $a(\eta) = \frac{1}{2}(1 + \cos\eta)$:

$$\longrightarrow \boxed{\chi = \sin^{-1}\left(\sqrt{\frac{8GM}{R_*^3}} \cdot \frac{r}{1 + \cos\eta}\right)} \quad \longleftarrow$$

where χ, r, η all <u>change</u> w.r.t. time

- For the next part of the problem assume $R_* = 5GM$ is the initial radius:

- A Schwarzchild black hole's event horizon is a null surface that is "generated" by null geodesics whose coordinate locations are $r = 2GM$ for all time. The event horizon of a black hole that forms in collapse is "generated" by the null geodesic that begins at the star's <u>center</u> and reaches the surface just as the surface passes through $r = 2GM$; at that point, by Birkhoff's theorem, this horizon "generator" will remain at $r = 2GM$ for all time.

$\boxed{f}$ · Determine the time τ at which the horizon generator leaves the <u>center</u> of the star:

- The parametric solution lets us write the space-time as:

$$ds^2 = a^2(\eta) R_0^2 \left(-d\eta^2 + d\chi^2 + \sin^2(\chi) d\Omega^2\right)$$

- Therefore, an outward propagating null geodesic obeys $d\chi / d\eta = 1$

$$\rightarrow \Delta\chi = \Delta\eta \longrightarrow \chi_f - \chi_i = \eta_f - \eta_i$$

· Since the null geodesic starts at the center that implies that $\chi_i = 0$:

$$\longrightarrow \eta_f = \eta_i + \chi_f \quad \text{or} \quad \eta_i = \eta_f - \chi_f$$

· We want to solve for η_i and convert it to τ_i: To do this we must first find η_f and χ_f. Let's start with η_f:

· Use the fact that $r = \frac{R_*}{2}(1 + \cos\eta)$

$$\longrightarrow r_f = 2GM = \frac{5GM}{2}(1 + \cos\eta_f)$$

$$\longrightarrow \eta_f = \cos^{-1}(-1/5) \ \checkmark$$

· Now find χ_f using our formula from part $\boxed{e}$:

$$\chi_f = \sin^{-1}\left(\sqrt{\frac{8GM}{5^3G^3M^3}} \cdot \frac{2GM}{1 + \cos(\eta_f)}\right)$$

$$\downarrow$$

$$\chi_f = \sin^{-1}\left(\left(\frac{2}{5}\right)^{3/2} \cdot \frac{2}{1 - 1/5}\right) = \sin^{-1}(0.632)$$

· So we get: $\quad \eta_i = \cos^{-1}\left(-\frac{1}{5}\right) - \sin^{-1}(0.632)$

$$\longrightarrow \eta_i \approx 1.088$$

· Now we plug this into our formula:

$$\tau_i = \sqrt{\frac{R_*^3}{8GM}} \left(1.088 + \sin(1.088)\right)$$

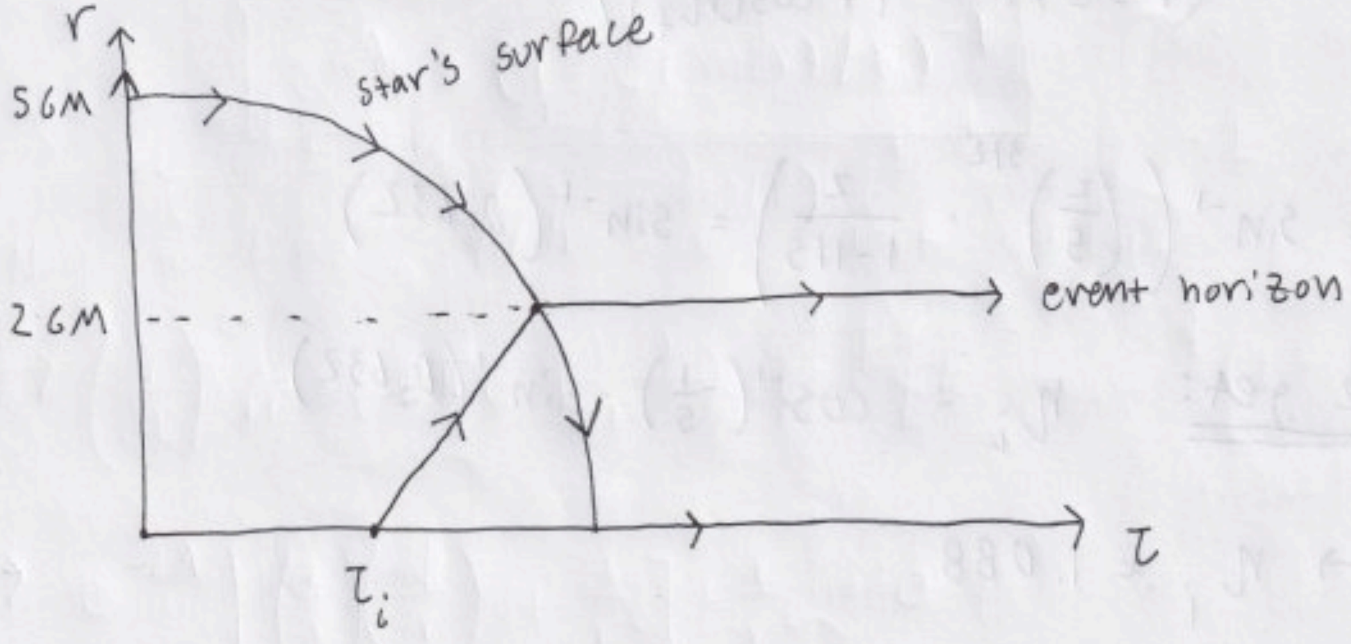
$$\downarrow$$

$$\tau_i = \left(\frac{5}{2}\right)^{3/2} (1.974)\ GM$$

$$\downarrow$$

$\boxed{\tau_i \approx 7.802\ GM}$ is the time at which the null geodesic leaves the center of the star + moves outward "generating" the event horizon...

$\boxed{9}$. Draw the star's surface + event horizon on a spacetime diagram: (Null geodesics always travel on 45° angles)



$\boxed{2}$. Consider a static spherical star cluster in which all stars move in circular orbits. Approximate the stars as pressureless dust, and write the normal Schwarzchild metric in the form:

$$ds^2 = -e^{2\phi} dt^2 + e^{2\Lambda} dr^2 + r^2 d\Omega^2$$

$$\updownarrow$$

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{(1 - 2GM/r)} + r^2 d\Omega^2$$

$\boxed{a}$. Find $e^{2\Lambda}$ and $d\phi/dr$ as functions of $m = \int_0^r 4\pi \rho(r) r^2 dr$ where we assume a continuum treatment for $\rho = \rho(r)$:

- The approach to this problem is to first:

$\rightarrow$ given metric compute Γ's

$\rightarrow$ given Γ's compute Ricci tensor

$\rightarrow$ given Ricci Tensor compute Ricci Scalar

$\rightarrow$ Use R + $R_{\alpha\beta}$ to find $G_{\alpha\beta}$

$\rightarrow$ Equate $G_{\alpha\beta}$ to $T_{\alpha\beta}$ using EFES.

- In lecture we already computed G_{tt}, G_{rr}, $G_{\theta\theta}$, and $G_{\varphi\varphi}$ for this line element so I will

"Steal" those results:

$$G_{tt} = -\frac{1}{r^2} \cdot \frac{d}{dr}\left[r\left(1 - e^{-2\Lambda}\right)\right]$$

$$G_{rr} = e^{-2\Lambda}\left[\frac{2}{r} \cdot \frac{d\phi}{dr} + \frac{1}{r^2}\right] - \frac{1}{r^2}$$

- The key difference now is we assume a pressureless system such that:

$$[T_{\nu r}] = \text{diag}(\rho, 0, 0, 0)$$

- By equating terms in the 2 equivalent Schwarzschild line elements we can find an expression for $e^{2\Lambda}$ quickly:

$$\boxed{e^{2\Lambda} = (1 - 2Gm(r)/r)^{-1}}$$

- Now to find $d\phi/dr$ we start with:

$$G_{rr} = 8\pi G\, T_{rr} = \emptyset$$

$$\rightarrow e^{-2\Lambda}\left[\left(\frac{2}{r}\right)\left(\frac{d\phi}{dr}\right) + \frac{1}{r^2}\right] - \frac{1}{r^2} = 0$$

$$\longrightarrow \left(\frac{2}{r}\right)\left(\frac{d\phi}{dr}\right) + \frac{1}{r^2} = \frac{e^{2\Lambda}}{r^2} = \frac{1}{r^2(1-2Gm(r)/r)}$$

$$\longrightarrow \frac{d\phi}{dr} = \left(\frac{\cancel{r}}{\cancel{2}}\right)\left(\frac{\cancel{1}-(\cancel{1}-\cancel{2}GM(r)/\cancel{r})}{r^2(1-2Gm(r)/r}\right)$$

$$\Rightarrow \boxed{\frac{d\phi}{dr} = \frac{Gm(r)}{r(r-2GM(r))}}$$

This is exactly equivalent to what was derived in lecture but with $P \to 0$

b . Define an appropriate effective potential $V_{eff}(r)$. Use it to determine the energy per unit mass $\hat{E}$ and angular momentum per unit mass $\hat{L}$ of a star in the cluster. Your answer should be expressed in terms of r, $m(r)$, $\phi(r)$. Determine the orbitall frequency $\Omega \equiv d\phi/dt = (d\phi/d\tau)/(dt/d\tau)$:

· The 4-momentum of one of these stars is given by

$p^\nu = m\left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, \frac{d\varphi}{d\tau}\right)$ by assuming no polar velocities...

$$P_t = -E = g_{tt}\,p^t = -me^{2\phi}\frac{dt}{d\tau}$$

where we are careful with the notational convention that $\phi = \phi(r)$ and φ is a coordinate...

$$p_\varphi = L_z = g_{\varphi\varphi} p^\varphi = m r^2 \frac{d\varphi}{d\tau}$$

· These relations imply that:

$$\frac{dt}{d\tau} = e^{-2\phi}\hat{E} \quad \text{where} \quad \hat{E} \equiv E/m$$

$$\frac{d\varphi}{d\tau} = \hat{L}/r^2 \quad \text{where} \quad \hat{L} \equiv L_z/m$$

· Now enforce $\vec{p}\cdot\vec{p} = -m^2$

$$\longrightarrow -m^2 = g_{\alpha\beta} p^\alpha p^\beta = g_{rr} p^r p^r + g_{\varphi\varphi} p^\varphi p^\varphi + g_{tt} p^t p^t$$

$$\longrightarrow -\cancel{m}^2 = -\cancel{m}^2 e^{2\phi}\left(\frac{dt}{d\tau}\right)^2 + \cancel{m}^2\left(1 - \frac{2Gm(r)}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + \cancel{m}^2 r^2 (d\varphi/d\tau)^2$$

$$\longrightarrow -e^{-2\phi}\hat{E}^2 + \frac{dr^2/d\tau^2}{(1-2Gm(r)/r)} + \frac{\hat{L}^2}{r^2} = -1$$

$$\longrightarrow \left(\frac{dr}{d\tau}\right)^2 = \left(1 - \frac{2Gm(r)}{r}\right) e^{-2\phi}\left[\hat{E}^2 - e^{2\phi}\left(1 + \frac{\hat{L}^2}{r^2}\right)\right]$$

· We define the last term as the effective potential:

$$\rightarrow \boxed{V_{eff}(r) = e^{2\phi(r)}\left(1 + \hat{L}^2/r^2\right)}$$

- If we use $\phi(r) \equiv \frac{Gm(r)}{r}$ then the pre-factor in the equation for $dr/d\tau$ goes to 1 and we get:

$$\frac{dr}{d\tau} = \pm\sqrt{\hat{E}^2 - e^{2\phi}\left(1 + L^2/r^2\right)}$$

$$\rightarrow \frac{dr}{d\tau} = \pm\sqrt{\hat{E}^2 - V_{eff}(r)}$$

- To enforce $\mathrm{Im}\{dr/d\tau\} = 0$, we need that $\hat{E} > \sqrt{V_{eff}(r)}$

- For a [redacted] circular orbit we need both:

$$dr/d\tau = 0 \longrightarrow \sqrt{V_{eff}(r)} = \hat{E}$$

- and -

$$\partial_r V_{eff}(r) = 0$$

- Start by taking the partial:

$$\partial_r V_{eff} = 2\frac{d\phi}{dr}e^{2\phi}\left(1 + \frac{L^2}{r^2}\right) - 2e^{2\phi}L^2/r^3 = 0$$

$$\longrightarrow \phi'(1 + L^2/r^2) = L^2/r^3$$

$$\longrightarrow \phi'(r^3 + L^2 r) = L^2$$

$$\longrightarrow r\phi'(r^2 + L^2) = L^2$$

$$\longrightarrow L^2(1 - r\phi') = r^3\phi'$$

$$\longrightarrow \boxed{\hat{L}^2 = \frac{r^3\phi'}{1 - r\phi'}}$$ for a circular orbit ...

· Now plug this L^2 back into V_{eff} + take a square root to find E:

$$V_{eff} = e^{2\phi}(1 + L^2/r^2)$$

$$\longrightarrow V_{eff} = e^{2\phi}\left(1 + \frac{r\phi'}{1 - r\phi'}\right) = \frac{e^{2\phi}}{1 - r\phi'}$$

$$\longrightarrow \boxed{\hat{E} = \frac{e^{\phi}}{\sqrt{1 - r\phi'}}}$$ for a circular orbit ...

· Now find $\Omega = \dfrac{d\varphi/d\tau}{dt/d\tau} = \dfrac{e^{2\phi}\tilde{L}}{r^2\hat{E}}$

· plug in $\hat{E}$ and $\hat{L}^2$ values :

$$\rightarrow \Omega = \left(\frac{e^{2\phi}}{r^2}\right)\left(\frac{\sqrt{r^3\phi'}}{\sqrt{\cancel{1-r\phi'}}}\right)\left(\frac{\sqrt{\cancel{1-r\phi'}}}{\sqrt{e^{2\phi}}}\right)$$

$$\downarrow$$

$$\Omega = \sqrt{\frac{e^{2\phi}\,\phi'}{r}}$$

- Now use $e^{2\phi} \approx 1 - \frac{2Gm}{r}$ compared to other form of schwarzchild line element...

and $\frac{d\phi}{dr} = \phi' = \frac{Gm(r)}{r(r-2Gm(r))}$ derived previously...

$$\rightarrow \Omega = \sqrt{\frac{Gm\,(1-2Gm/r)}{r^2(r-2Gm)}\cdot\frac{r}{r}}$$ use this to cancel terms

$$\rightarrow \boxed{\Omega = \sqrt{Gm/r^3} \text{ as in Newtonian Gravity}}$$

$\boxed{c}$ · Use V_{eff} to analyze the stability of circular orbits. In order for an orbit to be stable it must be located at a concave up minimum value of V_{eff} i.e. $\partial_r^2 V_{eff} > 0$. In lecture we derived the marginally stable orbit for

$$\partial_r^2 V_{eff} = 0 \longrightarrow r_{MS} = 6GM$$

· Therefore, all stable orbits must obey:

$$\boxed{\frac{Gm(r)}{r} < \frac{1}{6}}$$

locally at radius r + mass function $m = m(r)$ ...

[d] · Apply the above results to a homogeneous cluster of total mass M and radius R. "Homogeneous" implies $\rho \rightarrow$ constant and $m(r) =$ [redacted] $M(r/R)^3$ for $r \leq R$. You will need to use this to solve for $\phi(r)$. Find the maximum value of GM/R if <u>all</u> orbits are to be stable:

· We want $\frac{Gm(r)}{r} < \frac{1}{6}$ for all stars in this cluster... even for max $(m(r))$ the max value that $m(r)$ can take on. If the condition holds true for max $(m(r)/r)$ then it should hold true for <u>all</u> orbits:

$$\frac{Gm(r)}{r} = \frac{GMr^3}{rR^3} = \frac{GMr^2}{R^3} \quad \text{and} \ldots$$

$$\text{Max}\left(\frac{GMr^2}{R^3}\right) = \frac{GMR^2}{R^3} = \frac{GM}{R}$$

· So we require globally that $\boxed{\frac{GM}{R} < \frac{1}{6}}$ in order for $\frac{Gm(r)}{r} < \frac{1}{6}$ for all orbits with radius "r" $\forall$...

$\boxed{e}$. For the cluster with maximal GM/R, compute the redshift of photons emitted from the cluster's surface + from its center:

· I was not able to find a formula for redshift that applies here from Prof. Hughes' notes, but from wikipedia we found that the redshift of a Schwarzchild geometry can be computed as:

$$1+z = \sqrt{\frac{g_{tt}(\text{receiver})}{g_{tt}(\text{source})}}$$ we will assume the receiver is far away s.t. we can use the exterior schwarzchild solution:

$$g_{tt}(\text{receiver}) \approx 1 - \frac{2GM}{R} \approx 1 \quad \text{since } R \gg GM$$

$$\rightarrow 1+z \approx 1/\sqrt{g_{tt}(\text{source})}$$

- We must use the interior Schwarzschild solution for $g_{tt}(\text{source}) = e^{2\phi}$ in general though.
- So we must find ϕ:

$$\phi = \int \frac{d\phi}{dr}\, dr = \int \frac{G m(r)\, dr}{r^2(1 - 2Gm(r)/r)}$$

$$= \int \frac{G M r^3\, dr}{r^2 R^3\left(1 - \frac{2GMr^3}{rR^3}\right)} = \int \frac{GMr\, dr}{R^3(1 - 2GMr/R^3)}$$

Integral calculator $\rightsquigarrow$ $= -\frac{1}{4} \ln\left|1 - \frac{2GMr^2}{R^3}\right| + C$

- And we find the constant of integration by enforcing:

$$e^{2\phi(r=R)} = (1 - 2GM/R)$$

$$\rightarrow \phi(R) = \ln\left(\sqrt{1 - 2GM/R}\right)$$

set equal to our result at $r=R$:

$$\ln\left(\sqrt{1 - 2GM/R}\right) + \ln\left(\left(1 - 2GMR^2/R^3\right)^{1/4}\right) = C$$

$$\longrightarrow C = \ln\left(\left(1 - \frac{2GM}{R}\right)^{3/4}\right)$$

· So in general:

$$\phi(r) = \ln\left((1 - 2GM/R)^{3/4}\right) - \frac{1}{4}\ln\left(\left(1 - \frac{2GMr^2}{R^3}\right)\right)$$

$$= \frac{1}{2}\ln\left[\frac{(1-2GM/R)^{3/2}}{(1-2GMr^2/R^3)^{1/2}}\right]$$

$$\longrightarrow g_{tt}(r) = e^{2\phi(r)} = \frac{(1-2GM/R)^{3/2}}{(1-2GMr^2/R^3)^{1/2}}$$

· So compute $g_{tt}(r=0)$ at the center. for $\frac{GM}{R} = \frac{1}{6}$

$$g_{tt}(r=0) = \frac{(2/3)^{3/2}}{(1^{1/2})} = (2/3)^{3/2}$$

$$\longrightarrow 1 + z_{center} = (2/3)^{-3/4} \approx 1.355$$

$$\longrightarrow \boxed{z_{center} \approx 0.355}$$

· Also $g_{tt}(r=R) = \frac{(2/3)^{3/2}}{(2/3)^{1/2}} = (2/3)$

$$\longrightarrow 1 + z_{edge} = (2/3)^{-1/2} \approx 1.225$$

$$\longrightarrow \boxed{z_{edge} \approx 0.225}$$

- The problem states modern day measurements for the redshift of quasars is $z \approx 6.5$ which implies there is some other phenomenon at play since our results were of the order $z \approx 0.3$ ✓

[3] Numerical studies of black hole orbits

- In lecture we found that the following equations govern the motion of a test body moving around a black hole:

$$\left(\frac{dr}{d\tau}\right)^2 = \hat{E}^2 - V_{eff}(r) \quad \text{where}$$

$$V_{eff}(r) = \left(1 - \frac{2GM}{r}\right)\left(1 + \frac{\hat{L}^2}{r^2}\right)$$

$$d\varphi/d\tau = \hat{L}/r^2 \quad ; \quad dt/d\tau = \hat{E}/(1 - 2GM/r)$$

- In this exercise we will numerically integrate these equations to study some interesting orbits. The equation for "r" is tricky, you must try taking an additional derivative of both sides + rearranging ...

- It is useful to work in units where $GM = 1$ implying $r \to r/GM$, $t \to t/GM$, $\tau \to \tau/GM$, $\hat{L} \to \hat{L}/GM$ ...

$\boxed{a}$. With this choice of $GM = 1$, what are the basic units of time + length if $M = 10$ solar masses?

$$M_{solar} \approx 1.99 \times 10^{31}\ kg$$

$$G \approx 6.67 \times 10^{-11}\ m^3 \cdot kg^{-1} \cdot s^{-2}$$

$$\boxed{\begin{aligned}[\text{length}] &= \frac{MG}{c^2} \approx 1.47 \times 10^4\ m \\ [\text{time}] &= MG/c^3 \approx 4.91 \times 10^{-5}\ sec\end{aligned}}$$

· for parts $\boxed{b}$, $\boxed{c}$, $\boxed{d}$, and $\boxed{e}$; see the attached Jupyter Notebook ...